Joe Dinius
HW 3
MATH 534A
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1. Tu, 8.1

   Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

   $$(u, v, w) = F(x, y) = (x, y, xy).$$

   Compute $F_*(\partial/\partial x)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial w$.

   Solution. Start with the relation

   $$\begin{aligned} F_*(\frac{\partial}{\partial x})u &= (a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w})u \\ &= a. \end{aligned}$$

   Similar results follow when evaluating the differential at v, w. From these results, we can find a, b, c.

   $$\begin{aligned} a &= F_*(\frac{\partial}{\partial x})u = \frac{\partial}{\partial x}(u \circ F) = \frac{\partial}{\partial x}x = 1 \\ b &= F_*(\frac{\partial}{\partial x})v = \frac{\partial}{\partial x}(v \circ F) = \frac{\partial}{\partial x}y = 0 \\ c &= F_*(\frac{\partial}{\partial x})w = \frac{\partial}{\partial x}(w \circ F) = \frac{\partial}{\partial x}(xy) = y \end{aligned}$$

   Therefore,

   $$F_*(\frac{\partial}{\partial x}) = \frac{\partial}{\partial u} + y\frac{\partial}{\partial w}.$$

   □

2. Tu, 8.3

   Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the open set

   $$U = \mathbb{R}^2 \setminus \{(x, 0) | x \geq 0\}.$$

   On U the polar coordinates r, θ are uniquely defined by

   $$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta,\ r > 0,\ 0 < \theta < 2\pi. \end{aligned}$$

   Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

Solution. By the chain rule,

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}\end{aligned}$$

and so,

$$\begin{aligned}\frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\end{aligned}$$

□

3. Tu, 8.9

   Let $X_1, \ldots, X_n$ be n vector fields on an open subset U of a manifold of dimension n. Suppose that at $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ about p such that $(X_i)_p = (\partial/\partial x^i)_p$ for all $i = 1, \ldots, n$.

   Solution. Let $(V, y^1, \ldots, y^n)$ be a chart about p. Suppose $(X_j)_p = \sum_i a^i{}_j \partial/\partial y^i|_p$. Since the $(X_j)_p$ are linearly independent, the matrix $A = [a^i{}_j]$ is non-singular. Define a new coordinate system by the following ($i = 1, \ldots, n$)

$$y^i = \sum_{j=1}^{n} a^i{}_j x^j.$$

   By the chain rule, we have that

$$\begin{aligned}\frac{\partial}{\partial x^j} &= \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}\\ &= \sum_i a^i{}_j \frac{\partial}{\partial y^i}\\ &= (X_i)_p \text{ at the point } p.\end{aligned}$$

□

4. Tu, 9.3

   Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \;\; z = xy,$$

   in $\mathbb{R}^3$ a C^∞ manifold? Prove your answer.

Proof. Yes, the solution set in $\mathbb{R}^3$ is a C^∞ manifold. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$(u, v) = F(x, y, z) = (x^3 + y^3 + z^3, xy - z).$$

Take S to be the level set $F^{-1}(1, 0)$. The Jacobian of F is then

$$J(F) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ y & x & -1 \end{bmatrix}$$

Check the 2×2 minors of $J(F)$. Where the determinant of these minors is 0, we have critical points. The determinants of the minors

$$\begin{vmatrix} 3x^2 & 3y^2 \\ y & x \end{vmatrix} = 0, \quad \begin{vmatrix} 3x^2 & 3z^2 \\ y & -1 \end{vmatrix} = 0$$

give the conditions that $y = \pm x$ and $x^2 + z^2 y = 0$. Taking the first condition and putting it into the second yields that $(x, y, z) = (0, 0, 0)$ for the determinant of the minors to be 0. But the point $(0, 0, 0)$ doesn't satisfy the condition that $x^3 + y^3 + z^3 = 1$. Therefore there are no critical points of F on S. Therefore, by the regular level set theorem, S is a regular submanifold of dim 1 in $\mathbb{R}^3$. Hence, S is a C^∞ manifold since F is a smooth map. □

5. Tu, 9.5

   Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$,

   $$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\},$$

   is a regular submanifold of $\mathbb{R}^3$.

   Solution. First, we need to find an adapted chart of $\mathbb{R}^3$ to the graph $\Gamma(f)$. Try the following

   $$(U, \phi) = (\mathbb{R}^3, x, y, z - f(x, y)).$$

   On the set $U \cap \Gamma(f)$, ϕ takes x, y to themselves. On this intersection, the only component allowed is $z = f(x, y)$. Therefore, ϕ takes the third component z to $f(x, y) - f(x, y) = 0$. This means that ϕ is defined by the vanishing of one coordinate and therefore, by Definition 9.1, $\Gamma(f)$ is a regular submanifold. □

6. Tu, 9.9

   A C^∞ map $f : N \to M$ is said to be *transversal* to a manifold $S \subset M$ if for every $p \in f^{-1}(S)$,

   $$f_*(T_pN) + T_{f(p)}S = T_{f(p)}M.$$

   (If A and B are subspaces of a vector space, their sum $A+B$ is the subspace consisting of all $a + b$ with $a \in A$ and $b \in B$. The sum need not be a direct sum.) The goal of this exercise is to prove the *transversality theorem*: if a C^∞ map $f : N \to M$ is transversal to a regular submanifold S of codimension k in M, then $f^{-1}(S)$ is a regular submanifold of codimension k in N.

When S consists of a single point c, transversality of f to S simply means that $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is especially useful in giving conditions under which the intersection of two submanifolds is a submanifold.

Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z(x^{m-k+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = (x^{m-k+1}, \ldots, x^m).$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.

Solution.

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(S) &= f^{-1}(U \cap S) \\ &= f^{-1}(g^{-1}(0)) \\ &= (g \circ f)^{-1}(0) \end{aligned}$$

□

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

Solution. Let $p \in f^{-1}(U) \cap f^{-1}(S)$. The pushforward $g_*(T_{f(p)}S) = 0$, and because $g : U \to \mathbb{R}^k$ is a projection $g_*(T_{f(p)}M) = T_0(\mathbb{R}^k)$. Apply g_* to the transversality equation and you get

$$g_* f_*(T_pN) = g_*(T_{f(p)}M) = T_0(\mathbb{R}^k).$$

Hence, $g \circ f : f^{-1}(U) \to \mathbb{R}^k$ is a submersion at p. Since p was an arbitrary point of $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$, the set $f^{-1}(U \cap S)$ is a regular level set of $g \circ f$. □

(c) Prove the transversality theorem.

Proof. By the regular level set theorem, $f^{-1}(U) \cap f^{-1}(S)$ is a regular submanifold of $f^{-1}(U) \subset N$. Thus, every point $p \in f^{-1}(S)$ has an adapted chart relative to $f^{-1}(S)$ in N. This completes the proof. □

7. Tu, 11.2

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point. (*Hint*: Use Corollary 11.9 and the connectedness of $\mathbb{R}^m$.)

Solution. If $f : N \to \mathbb{R}^m$ has no critical points, then the differential $f_{*,p}$ would be surjective for every $p \in N$. If the differential is surjective, then f would be a submersion. By Corollary 11.9, a submersion is an open map, meaning that $f(N)$ would be open in $\mathbb{R}^m$. This causes a problem, since the continuous image of a compact set is compact and any compact subset of $\mathbb{R}^n$ is closed and bounded. Therefore, $f(N)$ is a nonempty proper closed subset of $\mathbb{R}^m$. But, since $\mathbb{R}^m$ is connected, it cannot have nonempty subsets that are both open and closed.

Therefore, a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point.

8. Tu, 11.3

On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \text{ and } v(a, b, c) = b.$$

So, the derivations $\partial/\partial u|_p$ and $\partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p\mathbb{R}^3$. Thus,

$$\begin{aligned} i_*\left(\left.\frac{\partial}{\partial u}\right|_p\right) &= \alpha^1 \left.\frac{\partial}{\partial x}\right|_p + \beta^1 \left.\frac{\partial}{\partial y}\right|_p + \gamma^1 \left.\frac{\partial}{\partial z}\right|_p, \\ i_*\left(\left.\frac{\partial}{\partial v}\right|_p\right) &= \alpha^2 \left.\frac{\partial}{\partial x}\right|_p + \beta^2 \left.\frac{\partial}{\partial y}\right|_p + \gamma^2 \left.\frac{\partial}{\partial z}\right|_p, \end{aligned}$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

Solution. It can be shown that

$$\begin{aligned} i_*\left(\left.\frac{\partial}{\partial u}\right|_p\right) &= \frac{\partial}{\partial x} - \frac{a}{c}\frac{\partial}{\partial z} \\ i_*\left(\left.\frac{\partial}{\partial v}\right|_p\right) &= \frac{\partial}{\partial y} - \frac{b}{c}\frac{\partial}{\partial z} \end{aligned}$$

and therefore

$$\begin{aligned} (\alpha^1, \beta^1, \gamma^1) &= (1, 0, -a/c) \\ (\alpha^2, \beta^2, \gamma^2) &= (0, 1, -b/c) \end{aligned}$$

□

9. Tu, 11.4

Let $f : N \to M$ be a one-to-one immersion. Prove that if N is compact, then $f(N)$ is a regular submanifold of M.

Proof. Assume f is a one-to-one immersion and N compact. We know that a continuous map from a compact space to a Hausdorff space is closed. Since M is a manifold, we know it is Hausdorff and therefore we can conclude that f is closed. With this condition, f is a homeomorphism between N and M. Therefore, f is an embedding and, by Theorem 11.17, $f(N)$ is a regular submanifold of M. □